Math 2020B Tut 7

Q1: (Example of a non-retifiable curve)

Let $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(t) = [t, t\sin\frac{\pi}{t}]$. Show that $\int_{\frac{1}{n}}^{\frac{1}{n-1}} |\gamma'(t)|dt \geq \frac{2}{n}$.

Hence $\lim_{n\to\infty} \int_{\frac{1}{n}}^{1} |\gamma'(t)|dt = \infty$

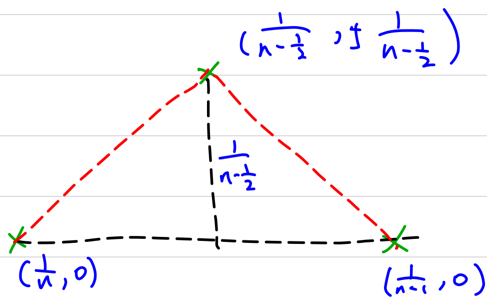
Ans: $\int_{\frac{1}{n}}^{\frac{1}{n-1}} |\gamma'(t)|dt = \text{length}(\gamma|_{[\frac{1}{n}, \frac{1}{n-1}]}) = \text{length}(\gamma|_{[\frac{1}{n}, \frac{1}{n-\frac{1}{2}}]}) + \text{length}(\gamma|_{[\frac{1}{n-\frac{1}{2}}, \frac{1}{n-1}]})$

$\geq |\gamma(\frac{1}{n-\frac{1}{2}}) - \gamma(\frac{1}{n})| + |\gamma(\frac{1}{n-1}) - \gamma(\frac{1}{n-\frac{1}{2}})|$

Now, $\gamma(\frac{1}{n-\frac{1}{2}}) = (\frac{1}{n-\frac{1}{2}}, \pm\frac{1}{n-\frac{1}{2}})$

$\gamma(\frac{1}{n}) = (\frac{1}{n}, 0)$

$\gamma(\frac{1}{n-1}) = (\frac{1}{n-1}, 0)$



Thus $\int_{\frac{1}{n}}^{\frac{1}{n-1}} |\gamma'(t)|dt$

$\geq \frac{1}{n-\frac{1}{2}} \cdot 2 \geq \frac{2}{n-1}$

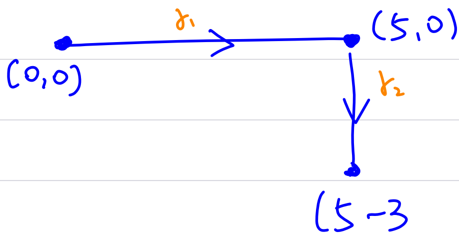
And $\int_{\frac{1}{n}}^{1} |\gamma'(t)|dt = \int_{\frac{1}{2}}^{1} |\gamma'(t)|dt + \int_{\frac{1}{3}}^{\frac{1}{2}} |\gamma'(t)|dt + \cdots + \int_{\frac{1}{n}}^{\frac{1}{n-1}} |\gamma'(t)|dt$

$= \frac{2}{2} + \frac{2}{3} + \cdots + \frac{2}{n}$

$\longrightarrow \infty$ as $n \to \infty$

Q2: C is the curve on the right

find $\int_C y\,dx + xy\,dy$



Ans: Step 1: $\gamma_1(t) = (t,0)$, $0 \le t \le 5$

$\gamma_2(s) = (5,-s)$, $0 \le s \le 3$

Step 2: $\gamma_1'(t) = (1,0)$, $r_2'(t) = (0,-1)$

$(y, xy) = (0,0)$ at $\gamma(t)$

$(y, xy) = (-s,0)$ at $\gamma(s)$

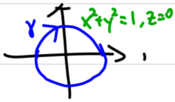
Step 3: $\int_C y\,dx + xy\,dy$

$= \int_{\gamma_1} y\,dx + xy\,dy + \int_{\gamma_2} y\,dx + xy\,dy$

$= \int_{\gamma_1} 0\,dx + 0\,dy + \int_0^3 (-s, -5s) \cdot (0,-1)\,ds$

$= 0 + \int_0^3 5s\,ds$

$= \frac{45}{2}$

Q3: 

, $\vec{F} = (x+y)\vec{i} - (x^2+y^2)\vec{j}$

Find the flow and Flux of $\vec{F}$ along γ.

Ans: $\gamma : [0, 2\pi] \to \mathbb{R}^3$, $\gamma(t) = (\cos(-t), \sin(-t), 0)$

$\gamma'(t) = (\sin(-t), -\cos(-t), 0)$

$F = (x+y)\vec{i} - \vec{j}$

$$\text{Flow} = \int_\gamma x+y \, dx - dy$$

$$= \int_0^{2\pi} (\cos(-t) + \sin(-t)) \, d\cos(-t) - d\sin(-t)$$

$$= \int_0^{2\pi} -\sin t \cos t + \sin^2 t + \cos t \, dt$$

$$= \int_0^{2\pi} -\frac{1}{2}\sin 2t + \frac{1}{2}(1 - \cos 2t) + \cos t \, dt$$

$$= \pi$$

$$\text{Flux} = \int_\gamma - dx - (x+y) \, dy$$

$$= \int_0^{2\pi} -d\cos(-t) - (\cos(-t) + \sin(-t)) \, d\sin(-t)$$

$$= \int_0^{2\pi} \sin t + \cos^2 t - \sin t \cos t \, dt$$

$$= \int_0^{2\pi} \sin t + \frac{1}{2}(1 + \cos 2t) - \frac{1}{2}\sin 2t \, dt$$

$$= \pi$$

Q4: Show that the Integral

$$\int_A^B z^2 dx + 2y + 2xz dz$$

is independent of the path taken from A to B.

Thoughts: "Want to write $(z^2, 2y, 2xz)$ as ∇f,

$$\nabla_x f = z^2 \Rightarrow f = xz^2 + g(y,z)$$

$$\nabla_y f = 2y \Rightarrow g(y,z) = y^2 + h(z)$$

$$\nabla_z f = 2xz \Rightarrow h(z) = \text{constant}"$$

Ans: Let $f(x,y,z) = xz^2 + y^2$, then $\nabla f(x,y,z) = (z^2, 2y, 2xz)$.

Thus $\int_A^B z^2 dx + 2ydy + 2xzdz = \int_A^B \nabla f \cdot \vec{T} ds$

$$= f(B) - f(A)$$

5 If C is a closed curve, show that

$$\int_C f\nabla g \cdot d\vec{r} = -\int_C g\nabla f\, d\vec{r}$$

Ans: $\int_C f\nabla g \cdot d\vec{r} + \int_C g\nabla f \cdot d\vec{r} = \int_C \nabla(fg) \cdot d\vec{r} = 0$

Q6: a) Consider the eqn $\tan\theta = \frac{y}{x}$ on $\mathbb{R}^2\setminus\{0\}$.

Note that the θ is not a (smooth) function on $\mathbb{R}^2\setminus\{0\}$.

However, show that $\nabla\theta$ is smooth on $\mathbb{R}^2\setminus\{0\}$.

b) find $\int_C \nabla\cdot\theta \cdot d\vec{r}$, where c is the positively oriented unit circle.

Ans: a) $\tan\theta = \frac{y}{x} \Rightarrow \begin{cases} \sec^2\theta \cdot \frac{\partial\theta}{\partial x} = -\frac{y}{x^2} \\ \sec^2\theta \cdot \frac{\partial\theta}{\partial x} = \frac{1}{x} \end{cases}$

$$\Rightarrow \begin{cases} \frac{\partial\theta}{\partial x} = \frac{1}{\sec^2\theta} \cdot \frac{-y}{x^2} = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2} \\ \frac{\partial\theta}{\partial y} = \frac{1}{\sec^2\theta} \cdot \frac{1}{x} = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} \end{cases}$$

$\Rightarrow \nabla\theta = \frac{1}{x^2+y^2}(-y\vec{i} + x\vec{j})$ is smooth on $\mathbb{R}^2\setminus\{0\}$

b) $\int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} -\sin t\, d\cos t + \cos t\, d\sin t$

$= \int_0^{2\pi} dt$

$= 2\pi$